## §3.2 Properties of determinants

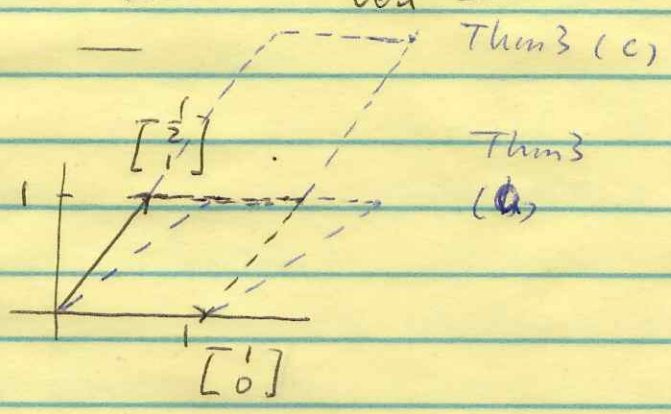
### Det as area or volume

Thm 9'. • $A$: $2\times 2$ matrix, then the "oriented" area of the parallelogram determined by the columns of $A$ $\rightsquigarrow$ $\det A$.

• $A$: $3\times 3$ matrix, —— vol —— parallelepiped —— $\det A$.

Exmp: $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

$\det A = 1$




Thm 3 (Row operations) $A$ square matrix

a) If $A \sim B$ add a multiple of a row to another row, then $\det A = \det B$ "Shear of the shape"

b) If $A \sim B$ interchange two rows, then $\det B = -\det A$.

c) If $A \sim B$ $k\times$ a row, then $\det B = k \det A$. "Scaling one side"

Exmp 1. $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

$\det A = ?$

|| transform it into an upper triangular mat.

Sol

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix}$$

$$= -1 \times 3 \times (-5) = 15$$

## det v.s. invertibility

**Thm 4** $A$ invertible $\Leftrightarrow$ $\det A \neq 0$.

**Exmp 4**
$$|A| = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} \overset{R4+R2}{=} \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

Cofactor expansion
$$= -2 \times \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} \overset{R2-3\times R1}{=} -2 \times \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= -2 \times (-5) \times \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -2 \times (-5) \times (-3) = -30$$

$A$ invertible

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## Column operations

**Thm 5** $\det A^T = \det A$

The Column operation version of Thm 3 also holds.

## Matrix products

**Thm 6** $\det(AB) = (\det A)(\det B)$

**Exmp** $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ $\det(AB) = ?$

$$|AB| = |A| \cdot |B| = (2\times 6 - 1\times 3) \times (4\times 2 - 1\times 3) = 9 \times 5 = 45$$

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## Linearity

$$|\vec{a}_1 \cdots \vec{a}_{j-1} \ \vec{b}_1 + \vec{b}_2 \ \vec{a}_{j+1} \cdots \vec{a}_n| = |\vec{a}_1 \cdots \vec{a}_{j-1} \ \vec{b}_1 \ \vec{a}_{j+1} \cdots \vec{a}_n| + |\vec{a}_1 \cdots \vec{a}_{j-1} \ \vec{b}_2 \ \vec{a}_{j+1} \cdots \vec{a}_n|$$